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Local and nonlocal solvable structures in the reduction of ODEs

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Abstract

Solvable structures, likewise solvable algebras of local symmetries, can be used to integrate scalar ordinary differential equations (ODEs) by quadratures. Solvable structures, however, are particularly suitable for the integration of ODEs with a lack of local symmetries. In fact, under regularity assumptions, any given ODE always admits solvable structures even though finding them in general could be a very difficult task. In practice, a noteworthy simplification may come by computing solvable structures which are adapted to some admitted symmetry algebra. In this paper we consider solvable structures adapted to local and nonlocal symmetry algebras of any order (i.e., classical and higher). In particular we introduce the notion of nonlocal solvable structure.

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1. Introduction

In recent years renewed interest in the classical works of Lie and Cartan led to further developments in the geometrical methods for the study of ordinary differential equations (ODEs). In particular, symmetry methods for ODEs have received increasing attention in the last 15 years. It is well known that if an ODE $\mathcal{E}$ admits a local symmetry, this can be used to reduce the order of $\mathcal{E}$ by one. This procedure, usually referred to as symmetry reduction method, is particularly useful when $\mathcal{E}$ is a k-order ODE whose local symmetries form a solvable k-dimensional Lie algebra. In this case, in fact, $\mathcal{E}$ can be completely integrated by quadratures [12, 23, 29]. However, finding all local symmetries of a given ODE $\mathcal{E}$ is not always possible. In fact local symmetries (classical or higher) of a k-order ODE in the unknown u are defined by the solutions of a linear partial differential equation (PDE) depending on the derivative of

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u up to the order $k-1$. Since the general solution of this PDE cannot be found unless one knows the general solution of $\mathcal{E}$, one usually only searches for particular solutions depending on the derivative of u up to order $k-2$. Therefore, in practice, $\mathcal{E}$ could not be reduced by quadratures if it does not admit a solvable k-dimensional algebra of such symmetries.

Nevertheless, one may encounter equations solvable by quadratures but with a lack of local symmetries. Examples of this kind, well known in recent literature [7–9, 15, 16, 21], seem to prove that local symmetries are sometimes inadequate and raise the question of whether a generalization of the notion of symmetry would lead to a more effective reduction method. For this reason, in the last few decades various generalizations of the classical symmetry reduction method have been proposed. Among these, in our opinion, special attention is deserved by the notions of *λ-symmetry* on the one hand and *solvable structure* on the other hand. Both notions, introduced in [3, 21], respectively, are suitable for a wide range of applications [4, 5, 9, 11, 14, 15, 20, 25, 26] and will be the starting point of our forthcoming discussion. In particular, we will provide a more effective reduction method which interconnects both notions.

The relevance of λ-symmetries is due to the fact that many equations, not possessing Lie point symmetries, admit λ-symmetries and these can be used to reduce the order exactly as in the case of standard symmetries. Despite their name, however, λ-symmetries are neither Lie point nor higher symmetries. As shown in [9], λ-symmetries of an ODE $\mathcal{E}$ can be interpreted as shadows of some nonlocal symmetries. In practice this means that, by embedding $\mathcal{E}$ in a suitable system $\mathcal{E}'$ determined by the function λ, any λ-symmetry of $\mathcal{E}$ can be recovered as a local symmetry of $\mathcal{E}'$. This interpretation of λ-symmetries has many advantages. First, it makes possible to give a precise geometric meaning to the λ-invariance property; second, it allows us to reinterpret the λ-symmetry reduction method as a particular case of the standard symmetry reduction.

Concerning solvable structures, they were introduced by Basarab–Horwath in [3] and further investigated in [4, 5, 17, 26]. The original purpose was the generalization of the classical result relating the integrability by quadratures of involutive distributions to the knowledge of a sufficiently large solvable symmetry algebra. The main point of this extension is that the vector fields assigning a solvable structure do not need to be symmetries; nevertheless they allow us to implement the reduction procedure. However, finding local solvable structures for a k-order ODE is not much easier than finding k-dimensional solvable algebras of local symmetries, if any. A noteworthy simplification, in general, may come by computing solvable structures adapted to the admitted symmetry algebras, if any. It follows that one certainly takes advantage of the presence of any kind of symmetry. In particular, one can include the nonlocal symmetries corresponding to λ-symmetries.

In this paper we consider solvable structures adapted to local and nonlocal symmetry algebras of any order (i.e., classical and higher), and we introduce the notion of nonlocal solvable structure.

This paper is organized as follows. In section 2, we collect all notations and basic facts we need on the geometry of differential equations. In particular we recall the definitions of local and nonlocal symmetries and also the interpretation of λ-symmetries as shadows of nonlocal symmetries. In section 3, we recall the results of [3, 4] in a form suitable to our further discussion on the integration of ODEs. Then, in section 4, we discuss our reduction scheme of ODEs via (higher order) local and nonlocal solvable structures. In particular, in this section, we discuss a practical method for the computation of solvable structures. Finally, in section 5 we collect some examples which provide insight into the application of the proposed reduction scheme.

2. Preliminaries

Throughout this paper we assume that the reader is familiar with the geometry of differential equations. Nevertheless, we collect here some notations and basic facts we need in this paper. The reader is referred to [6, 7, 13, 23, 24, 28, 29] for further details.

2.1. ODEs as submanifolds of jet spaces

Let M and E be smooth manifolds and $\pi : E \to M$ be a q-dimensional bundle. We denote by $\pi_k : J^k(\pi) \to M$ the k-order *jet bundle* associated with π and by $j_k(s)$ the k-order *jet prolongation* of a section s of π. Since in this paper we are only concerned with the case $\dim M = 1$, we assume that M and E have local coordinates x and $(x, u^1, \ldots, u^q)$, respectively. Correspondingly, the induced natural coordinates on $J^k(\pi)$ will be $\left(x, u^a_i\right)$, $1 \leqslant a \leqslant q, i = 0, 1, \ldots, k$, where the u^a_i's are defined by $u^a_i(j_\infty(s)) = \mathrm{d}^i(u^a(s))/\mathrm{d}x^i$, for any section s of π. Moreover, when no confusion arises, Einstein summation convention over repeated indices will be used.

The *k-order jet space* $J^k(\pi)$ is a manifold equipped with the smooth distribution $\mathcal{C}^k$ of tangent planes to graphs of k-order jet prolongations $j_k(s)$. This is the *contact (or Cartan) distribution of* $J^k(\pi)$.

In this framework a kth order system of differential equations can be regarded as a submanifold $\mathcal{E} \subset J^k(\pi)$ and any solution of the system is a section of π whose k-order prolongation is an integral manifold of the restriction $\mathcal{C}^k|_{\mathcal{E}}$ of the contact distribution to $\mathcal{E}$. In this paper we will deal only with (systems of) ordinary differential equations $\mathcal{E}$ which are in *normal form* and *not underdetermined*.

The natural projections $\pi_{h,k} : J^h(\pi) \to J^k(\pi)$, for any $h > k$, allow one to define the bundle of infinite jets $J^\infty(\pi) \to M$ as the inverse limit of the tower of projections $M \longleftarrow E \longleftarrow J^1(\pi) \longleftarrow J^2(\pi) \longleftarrow \ldots$.

The manifold $J^\infty(\pi)$ is infinite dimensional with induced coordinates $\left(x, u^a_i\right)$, $1 \leqslant a \leqslant q, i = 0, 1, \ldots,$ and the $\mathbb{R}$-algebra of smooth functions on $J^\infty(\pi)$ is defined as $\mathcal{F}(\pi) = \cup_l C^\infty(J^l(\pi))$. The set $D(\pi)$ of vector fields on $J^\infty(\pi)$ has the structure of a Lie algebra, with respect to the usual Lie bracket [10, 23, 29]. Moreover one can also define a *contact distribution* $\mathcal{C}$ on $J^\infty(\pi)$, generated by the *total derivative* operator

$$D_x = \partial_x + u^a_{i+1}\partial_{u^a_i}.$$

Note that $\mathcal{C}$ is the annihilator space of the *contact forms* $\theta^a_s = \mathrm{d}u^a_s - u^a_{s+1}\,\mathrm{d}x$.

Given a kth order differential equation $\mathcal{E} = \{F = 0\}$, the lth *prolongation of* $\mathcal{E}$ is the submanifold $\mathcal{E}^{(l)} := \left\{D^s_x(F) = 0 : s = 0, 1, \ldots, l\right\}$. Analogously, we define the *infinite prolongation* $\mathcal{E}^{(\infty)}$.

2.2. Local symmetries

The infinitesimal symmetries of $\mathcal{C}^k$ are called *Lie symmetries* of $J^k(\pi)$. These symmetries can be divided into two classes (see [23, 29]): *Lie point symmetries*, which are obtained by prolonging vector fields X on E, and *Lie contact symmetries*, which are obtained by prolonging symmetries X on $J^1(\pi)$.

By definition, symmetries of $\mathcal{C}^k$ which are tangent to $\mathcal{E}$ are called *classical symmetries of* $\mathcal{E}$.

An analogous geometric picture holds on the infinite jet spaces. However, in contrast to the case of finite-order jet spaces, symmetries of $\mathcal{C}$ cannot always be recovered by infinite prolongations of Lie symmetries. In fact, it can be proved that $X = \xi\partial_x + \eta_i^a\partial_{u_i^a}$ is an infinitesimal symmetry of $\mathcal{C}$ if and only if $\xi, \eta^a \in \mathcal{F}(\pi)$ are arbitrary functions and

$$\eta_i^a = D_x\left(\eta_{i-1}^a\right) - u_i^a D_x(\xi), \qquad \eta_0^a = \eta^a.$$

Hence, X is the infinite prolongation of a Lie point (or contact) symmetry iff ξ, η_0^a are functions on E (or $J^1(\pi)$, respectively).

On $J^\infty(\pi)$, symmetries X of $\mathcal{C}$ which are tangent to $\mathcal{E}^{(\infty)}$ are called *higher symmetries* of $\mathcal{E}$ and are determined by the condition $X(F)|_{\mathcal{E}^{(\infty)}} = 0$.

When working on infinite jet spaces, since D_x is a trivial symmetry of $\mathcal{C}$, it is convenient to gauge out from higher symmetries the terms proportional to D_x. This leads to considering only symmetries in the so-called *evolutive form*, i.e., symmetries of the form $X = D_x^i(\varphi^a)\partial_{u_i^a}, \varphi^a := \eta^a - u_1^a\xi$. The functions φ^a are called the *generating functions* (or characteristics) of X.

For the applications we are interested in, we only need to consider symmetries of $\mathcal{C}$ which are tangent to $\mathcal{E}^{(\infty)}$. This choice also turns out to be convenient since it noteworthily simplifies computations (see [2, 29] for more details and other aspects of ∞-jets theory).

As already remarked in the introduction, local symmetries (Lie or higher) of a k-order ODE

$$\mathcal{E} = \{u_k = f(x, u, u_1, \dots, u_{k-1})\}$$

in the unknown u are defined by the solutions of a linear PDE depending on the derivative of u up to the order $k - 1$. This linear PDE is usually called the determining equation and is obtained by writing in coordinates the symmetry condition $X(u_k - f)|_{\mathcal{E}^{(\infty)}} = 0$. Unfortunately, the general solution to this PDE cannot be found and one usually searches only for particular solutions, if any, which depends on the derivative of u up to the order $k - 2$. Hence, it is not unusual to run into ODEs for which we are unable to compute local symmetries.

2.3. Nonlocal symmetries and λ-symmetries

In recent literature, to overcome the difficulties due to a lack of local symmetries, various attempts have been made to find a more general notion of symmetry. Hence, some new classes of symmetries have been introduced. Among these generalizations there are those introduced by Muriel and Romero in [21] and known as λ-symmetries.

The notion of *nonlocal symmetry* which we use in this paper furnishes the conceptual framework to deal with many of these generalizations. We follow here the approach to nonlocality based on the theory of *coverings* (see [19] and also [29]). However, since we only deal with ODEs, our approach will be noteworthily simplified. The interested reader is referred to [29] for the general theory of coverings and nonlocal symmetries.

Definition 1. *Let $\mathcal{E}$ be a k-order ODE on a q-dimensional bundle π. We shall say that a smooth bundle $\kappa : \widetilde{\mathcal{E}} \to \mathcal{E}^{(\infty)}$ is a covering for $\mathcal{E}$ if the manifold $\widetilde{\mathcal{E}}$ is equipped with a one-dimensional distribution $\widetilde{\mathcal{C}} = \{\widetilde{\mathcal{C}}_p\}_{p\in\widetilde{\mathcal{E}}}$ and, for any point $p \in \widetilde{\mathcal{E}}$, the tangent mapping κ_* gives an isomorphism between $\widetilde{\mathcal{C}}$ and the restriction $\mathcal{C}|_{\mathcal{E}^{(\infty)}}$ of the contact distribution of $J^\infty(\pi)$ to $\mathcal{E}^{(\infty)}$.*

The dimension of the bundle κ is called the dimension of the covering and is denoted by $\dim(\kappa)$. Below we only consider the case $\dim(\kappa) = 1$.

Definition 2. *Nonlocal symmetries of $\mathcal{E}$ are the symmetries of the distribution $\widetilde{\mathcal{C}}$ of a covering $\kappa : \widetilde{\mathcal{E}} \to \mathcal{E}^{(\infty)}$.*

Here follows a coordinate description of the notion of covering and nonlocal symmetry. Since below we will restrict our attention only to nonlocal symmetries of scalar ODEs, in the following formulae we will restrict only to the case when $u^a = u$.

Let π be a trivial one-dimensional bundle over $\mathbb{R}$, with standard coordinates (x, u), and let

$$\mathcal{E} := \{u_k = f(x, u, u_1, \ldots, u_{k-1})\}. \tag{1}$$

Below we will consider only coverings where κ is a trivial bundle $\kappa : \mathcal{E}^{(\infty)} \times \mathbb{R} \to \mathcal{E}^{(\infty)}$, hence denoting by w the standard coordinate in $\mathbb{R}$, the distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}}$ is generated by the vector field

$$\widetilde{D}_x|_{\widetilde{\mathcal{E}}} = \overline{D}_x + H\partial_w, \tag{2}$$

where H is a smooth function on $\mathcal{E}^{(\infty)} \times \mathbb{R}$ and $\overline{D}_x = \partial_x + u_1\partial_u + \cdots + f\partial_{u_{k-1}}$ is the restriction to $\mathcal{E}^{(\infty)}$ of the total derivative operator on $J^\infty(\pi)$. Hence, the covering κ is determined by the system (see [9])

$$\mathcal{E}' := \{u_k = f, w_1 = H\}. \tag{3}$$

Nonlocal symmetries of $\mathcal{E}$ are symmetries of the vector field (2) and can be determined through a symmetry analysis of the system $(\mathcal{E}')^{(\infty)}$ on $J^\infty(\widetilde{\pi})$. Therefore nonlocal symmetries of $\mathcal{E}$ have the form

$$Y = \xi\partial_x + \eta_i\partial_{u_i} + \psi_i\partial_{w_i} \tag{4}$$

with $\eta_i = \widetilde{D}_x(\eta_{i-1}) - \widetilde{D}_x(\xi)u_i$ and $\psi_i = \widetilde{D}_x(\psi_{i-1}) - w_i\widetilde{D}_x(\xi)$.

An interesting example of nonlocal symmetry occurring in the literature is related to the notion of λ-symmetry for an ODE $\mathcal{E}$ (see [21]). Despite their name, in fact, λ-symmetries are neither Lie symmetries nor higher symmetries of $\mathcal{E}$. Nevertheless, as discussed in [9], λ-symmetries can be interpreted as shadows of nonlocal symmetries. More precisely, $\mathcal{E}$ admits a λ-symmetry X iff $\mathcal{E}' = \{u_k = f, w_1 = \lambda\}$, with $\lambda \in C^\infty(\mathcal{E})$, admits a (higher) symmetry with generating functions of the form $\varphi^\alpha = \mathrm{e}^w\varphi_0^\alpha(x, u, u_1, \ldots, u_{k-1})$, $\alpha = 1, 2$.

Since λ-symmetries are of great interest in the applications, and analogously their nonlocal counterparts, it is convenient to introduce the following.

Definition 3. *If a covering system $\mathcal{E}'$, defined by (3) with $H = \lambda$ and $\lambda \in C^\infty(\mathcal{E})$, admits a nonlocal symmetry Y with generating functions*

$$\varphi^\alpha = \mathrm{e}^w\varphi_0^\alpha(x, u, u_1, \ldots, u_{k-1}), \qquad \alpha = 1, 2 \tag{5}$$

then $\mathcal{E}'$ will be called a λ-covering for $\mathcal{E}$ defined by (1).

3. Solvable structures

In this section we will recall basic definitions and facts on (local) solvable structures in the form we need in our study. The reader is referred to [3, 4] for further details.

The original purpose of solvable structures was to generalize the classical result stating that knowledge of a solvable k-dimensional algebra of symmetries for an $(n - k)$-dimensional involutive distribution $\mathcal{D}$, on an n-dimensional manifold N, guarantees that $\mathcal{D}$ can be integrated by quadratures. Solvable structures are a generalization of solvable symmetry algebras which allows one to keep this classical result.

Since in this paper we restrict our attention to the integration of one-dimensional distributions, we will summarize the results of [3, 4] only in this case.

Given a one-dimensional distribution $\mathcal{D} = \langle Z\rangle$, on an n-dimensional manifold N, the definition of a solvable structure for $\mathcal{D}$ is the following.

Definition 4. *The vector fields* $\{Y_1, \ldots, Y_{n-1}\}$ *on* N *form a solvable structure for* $\mathcal{D} = \langle Z \rangle$ *if and only if, denoting* $\mathcal{D}_0 = \mathcal{D}$ *and* $\mathcal{D}_h = \langle Z, Y_1, \ldots, Y_h \rangle$*, the following two conditions are satisfied:*

(i) $\mathcal{D}_{n-1} = TN$;
(ii) $L_{Y_h}\mathcal{D}_{h-1} \subset \mathcal{D}_{h-1}, \forall\, h \in \{1, \ldots, n-1\}$.

In particular, given a solvable structure $\{Y_1, \ldots, Y_{n-1}\}$ for $\mathcal{D}$, one has the flag of integrable distributions,

$$\langle Z \rangle = \mathcal{D}_0 \subset \mathcal{D}_1 \subset \ldots \subset \mathcal{D}_{n-1} = TN \tag{6}$$

with $\mathcal{D}_{s-1} = \{[X, Y] : X, Y \in \mathcal{D}_s\}$, and $[X, Y]$ denoting the Lie bracket of X and Y.

The main difference with the definition of a solvable symmetry algebra of $\mathcal{D}$ is that the fields belonging to a solvable structure do not need to be symmetries for $\mathcal{D}$. This, of course, gives more freedom in the choice of the fields one can use in the integration of $\mathcal{D}$.

Remark 1. It is straightforward, by the definition above, that in principle a solvable structure for $\mathcal{D}$ always exists in a neighborhood of a non-singular point for $\mathcal{D}$. In fact, if $\{x^i\}$ is a local chart on N such that $Z = \partial_{x^1}$, one can simply consider the solvable structure generated by $\partial_{x^j}, j = 2, \ldots, n$. This structure is in particular an Abelian algebra of symmetries for $\mathcal{D}$. Nevertheless, for a given distribution $\mathcal{D}$, it is difficult to find explicitly such a local chart.

In the case of one-dimensional distributions, the main result of [3] can be stated as follows:

Proposition 1. *Let* $\{Y_1, \ldots, Y_{n-1}\}$ *be a solvable structure for a one-dimensional distribution* $\mathcal{D} = \langle Z \rangle$ *on an orientable* n*-dimensional manifold* N*. Then, for any given volume form* Ω *on* N*,* $\mathcal{D}$ *can be described as the annihilator space of the system of 1-forms* $\{\omega_1, \ldots, \omega_{n-1}\}$ *defined as*

$$\omega_i = \frac{1}{\Delta}(Y_1 \lrcorner \ldots \lrcorner \widehat{Y}_i \lrcorner \ldots \lrcorner Y_{n-1} \lrcorner Z \lrcorner \Omega), \tag{7}$$

where the hat denotes omission of the corresponding field, $\lrcorner$ *denotes the insertion operator (see [27]) and* Δ *is the function defined by*

$$\Delta = Y_1 \lrcorner Y_2 \lrcorner \ldots \lrcorner Y_{n-1} \lrcorner Z \lrcorner \Omega.$$

Moreover, one has that

$$\begin{aligned} \mathrm{d}\omega_{n-1} &= 0, \\ \mathrm{d}\omega_i &= 0 \quad \mathrm{mod}\{\omega_{i+1}, \ldots, \omega_{n-1}\} \end{aligned}$$

for any $i \in \{1, \ldots, n-2\}$.

The reader is referred to [3] or [4] for the proof.

It follows that, under the assumptions of this proposition, $\mathcal{D}$ can be completely integrated by quadratures (at least locally). In fact, in view of $\mathrm{d}\omega_{n-1} = 0$, locally $\omega_{n-1} = \mathrm{d}I_{n-1}$ for some smooth function I_{n-1}. Now, since $\mathrm{d}\omega_{n-2} = 0 \;\mathrm{mod}\{\omega_{n-1}\}$, ω_{n-2} is closed on the level manifolds of I_{n-1}. Then, iterating this procedure, it is possible to compute all the integrals $\{I_1, \ldots, I_{n-1}\}$ of $\mathcal{D}$ and eventually find its (local) integral manifolds in implicit form $\{I_1 = c_1, \ldots, I_{n-1} = c_{n-1}\}$.

Remark 2. If one knows a solvable structure, proposition 1 allows a complete reduction (or integration) of $\mathcal{D}$. However, if Y_1 is a symmetry of $\mathcal{D} = \langle Z \rangle$ and $Z, Y_1, Y_2, \ldots, Y_{n-1}$ is just a frame on M, then one can still define the forms $\{\omega_i\}$ and $\mathcal{D} = Ann(\omega_1, \ldots, \omega_{n-1})$. In this case, one cannot completely integrate $\mathcal{D}$, but just reduce its codimension by one. This partial reduction is a particular case of that described in the paper [1] and in terms of the forms ω_i's can be described as follows. Since Y_1 is a symmetry of $\mathcal{D}$, then it is also a symmetry of the Pfaffian system $\mathcal{I}$ generated by $\omega_1, \ldots, \omega_{n-1}$. It follows that locally the quotient of M by the action induced by Y_1 is a manifold $\overline{M}$ naturally equipped with a Pfaffian system $\overline{\mathcal{I}}$. Since $Y_1 \lrcorner \omega_j = \delta_{1j}$, one has that $\overline{\mathcal{I}} = \{\omega_2, \ldots, \omega_{n-1}\}$. Now the above-mentioned reduced distribution is just $\overline{\mathcal{D}} = Ann(\overline{\mathcal{I}})$.

Proposition 1 can also be applied to the integration of ODEs. In fact, in view of the above discussion, one can think of an ODE $\mathcal{E}$ as a manifold equipped with the one-dimensional distribution $\mathcal{D} = \langle \overline{D}_x \rangle$. In particular, equation $\mathcal{E}$ defined by (1) can be seen as a $(k+1)$-dimensional submanifold of $J^k(\pi)$ naturally equipped with the volume form

$$\Omega = \mathrm{d}x \wedge \mathrm{d}u \wedge \ldots \wedge \mathrm{d}u_{k-1}, \tag{8}$$

where $\wedge$ denotes the usual wedge product.

Then, by applying proposition 1 to ODEs one readily gets the following.

Proposition 2. *Let $\{Y_1, \ldots, Y_k\}$ be a solvable structure for the one-dimensional distribution $\mathcal{D} = \langle \overline{D}_x \rangle$ on $\mathcal{E}$ defined by (1). Then $\mathcal{E}$ is integrable by quadratures and the general solution of $\mathcal{E}$ can be obtained in an implicit form by subsequently integrating the system of 1-forms $\omega_k, \ldots, \omega_1$, in the given order.*

Remark 3. The facts observed above in remark 2 apply also to the case of ODEs. As an example, consider the ODE $\mathcal{E} := \{u_2 = -u/x^2\}$. This ODE admits the classical symmetry $x\partial_x$. The reduced equation $\overline{\mathcal{E}}$ is equipped with the Pfaffian system $\overline{\mathcal{I}}$ generated by ω_2, or equivalently by $\Delta\omega_2$. Since $\Delta\omega_2$ is a 1-form on the quotient $\overline{\mathcal{E}}$, then it can be written only in terms of the invariants of Y_1. Now, since the invariants of Y_1 are just $I_1 = u$, $I_2 = xu_x$, one can check that $\overline{\mathcal{I}} = \{\mathrm{d}I_2 - (I_2 - I_1)\mathrm{d}I_1/I_2\}$. Then integral manifolds of $\overline{\mathcal{I}}$ are exactly the solutions of the reduced ODE $\mathrm{d}I_2/\mathrm{d}I_1 = (I_2 - I_1)/I_2$ obtained by using the standard symmetry reduction method [23].

4. New applications of solvable structures to the integration of ODEs

In this section we discuss our new reduction scheme for ODEs.

As remarked above, solvable structures have been introduced to generalize the standard symmetry reduction method. Finding solvable structures, however, is not always easy and in practice a noteworthy simplification may come by computing solvable structures adapted to an admitted symmetry algebra $\mathcal{G}$, if any. In fact, as discussed below, under this assumption the determining equation of a solvable structure extending $\mathcal{G}$ becomes more affordable. Moreover, since the language of covering allows one to handle nonlocal symmetries just like local symmetries, we can take advantage of solvable structures adapted also to nonlocal symmetries. In this paper we consider solvable structures adapted to any kind (local, higher and nonlocal) of symmetry algebras.

Now we first discuss the practical determination of solvable structures for ODEs. To this end, it is useful to have the following technical lemma which expresses the symmetry condition in the framework of multivector fields (see [10]).

Lemma 1. *Let $\mathcal{D} = \langle X_1, \ldots, X_r \rangle$ be an r-dimensional distribution on an n-dimensional manifold N. The vector field X is a symmetry of $\mathcal{D}$ iff*

$$L_X(X_i) \wedge X_1 \wedge X_2 \wedge \ldots \wedge X_r = 0, \qquad \forall\, i = 1, 2, \ldots, r. \tag{9}$$

In view of this lemma, in fact, one readily gets the following.

Proposition 3. *Let $\mathcal{E}$ be defined by (1), Ω be the volume form (8) and ω_i's defined by (7). The vector fields $\{Y_1, \ldots, Y_k\}$ on $\mathcal{E}$ determine a solvable structure for $\mathcal{D} = \langle \overline{D}_x \rangle$ iff the following two conditions are satisfied:*

(a) $Y_1 \lrcorner Y_2 \lrcorner \ldots \lrcorner Y_k \lrcorner \overline{D}_x \lrcorner \Omega \neq 0$;
(b) for any $s = 0, 1, \ldots, k-1$, defining $Y_0 = \overline{D}_x$, one has

$$L_{Y_{s+1}}(Y_j) \wedge Y_0 \wedge Y_1 \wedge \ldots \wedge Y_s = 0, \qquad \forall\, j = 0, 1, \ldots, s. \tag{10}$$

Proof.

(a) By using the notation of definition 4, one has that $\mathcal{D}_k = T\mathcal{E}$. Hence the vector fields $Y_1, \ldots, Y_k, \overline{D}_x$ are linearly independent and the conclusion follows.
(b) The result follows by proposition 1 and (9). □

Remark 4. By using the volume form Ω, conditions (10) can also be rewritten in the alternative form

$$L_{Y_{s+1}}(Y_j) \lrcorner (Y_0 \lrcorner Y_1 \lrcorner \ldots \lrcorner Y_s \lrcorner \Omega) = 0, \qquad \forall\, j = 0, 1, \ldots, s. \tag{11}$$

Equations (10) (or (11)) are the determining equations for the solvable structures $\{Y_1, \ldots, Y_k\}$ of $\mathcal{D} = \langle \overline{D}_x \rangle$. If one writes all these equations in coordinate form, it turns out that these form a system of

$$\sum_{s=0}^{k-1} (s+1) \binom{k+1}{s+2} = 1 - 2^{k+1} + (k+1)2^k$$

nonlinear first-order PDEs in the $(k+1)k$ unknown components of the fields $Y_1, \ldots, Y_k$. It is underdetermined when $k = 1, 2$ and overdetermined when $k \geqslant 3$. However, this system has a very special form. In fact, for $s = 0$ equation (10) gives a system S_0 of equations involving only the field Y_1. Then, after determining a solution Y_1 and substituting into (10), for $s = 1$ one gets another system S_1 of equations only for Y_2. It follows that, iterating this procedure, the determination of solvable structures just follows by the subsequent analysis of the linear systems $S_0, \ldots, S_{k-1}$.

Remark 5. In view of the definition of solvable structure, any solution Y_1 of S_0 is a symmetry of the vector field $\overline{D}_x$ and hence it is a symmetry of $\mathcal{E} \subset J^k(\pi)$. Nevertheless, it does not mean that it is a symmetry of the contact distribution $\mathcal{C}^k$ on $J^k(\pi)$. In fact Y_1 could also be an internal symmetry of $\mathcal{E}$. For the details on possible relations between external, internal and generalized symmetries the reader is referred to [2].

In view of remark 1, the determining equations for solvable structures always admit local solutions even though they could be hardly solvable in practice. However, a noteworthy simplification may occur when one only knows $\{Y_1, \ldots, Y_r\}$, $r \leqslant k-1$, and a complete system of joint invariants $\{\gamma_1, \ldots, \gamma_{k-r}\}$ for the distribution $\mathcal{D}_r$. In fact, using the same notation of definition 4 and proposition 1 one has the following.

Proposition 4. *Let $\mathcal{D} = \langle Z \rangle$ be a one-dimensional distribution and $\{Y_1, \ldots, Y_r\}$, $r \leqslant k-1$, be such that $L_{Y_h}\mathcal{D}_{h-1} \subset \mathcal{D}_{h-1}$ for any $h \in \{1, \ldots, r\}$. If one knows a complete system of joint invariants $\{\gamma_1, \ldots, \gamma_{k-r-1}\}$ for the distribution $\mathcal{D}_r$, then $\mathcal{D}$ is completely integrable by quadratures.*

Proof. Let $\{\gamma_i, g_j\}$ be a local coordinate chart adapted to the leaves of the distribution $\mathcal{D}_r = Ann\{d\gamma_1, \ldots, d\gamma_{k-r-1}\}$. It turns out that any field ∂_{γ_i} is a symmetry of $\mathcal{D}_r$, for $L_{\partial_{\gamma_i}}(d\gamma_j) = 0$, and hence $\{Y_1, \ldots, Y_r, \partial_{\gamma_1}, \ldots, \partial_{\gamma_{k-r-1}}\}$ form a solvable structure. The conclusion follows by a direct application of proposition 1. □

It is clear that, since the joint invariants $F \in C^\infty(\mathcal{E})$ for $\mathcal{D}_r$ are described by the overdetermined system

$$\overline{D}_x(F) = Y_i(F) = 0, \qquad \forall\, i = 1, \ldots, r,$$

the more fields Y_i one knows, the more it is feasible to find a fundamental set of invariants $\{\gamma_1, \ldots, \gamma_{k-r}\}$. In fact, the analysis of differential consequences of such an overdetermined system of PDEs often leads to the determination of the general solution.

The above facts, in particular, apply to the case when $\mathcal{E}$ has an r-dimensional solvable algebra[3] $\mathcal{G} = \langle Y_1, \ldots, Y_r\rangle_{\mathbb{R}}$ of local classical symmetries. In such cases, the problem of finding solvable structures which extends $\mathcal{G}$ is much easier and feasible if r is sufficiently large. A first simple implementation of this idea, even though with a different approach, has already been given in the paper [17]. In fact, in that paper it is shown how such a kind of completion together with the results of [3] may lead to the integration of a second-order ODE whose Lie algebra of point symmetries is one-dimensional.

However there is no real need to limit ourselves to classical local symmetries. In fact, in view of the above discussion, the determination of solvable structures can be noteworthy simplified if one preliminarily enlarges $\mathcal{G}$ by adjoining as many symmetries as possible. Moreover, as already observed in the previous sections, one often runs into ODEs with a lack of local symmetries. Hence, in such cases, to overcome the occurring difficulties in the reduction procedure one could also take advantage of the existence of nonlocal symmetries.

However, as discussed above, nonlocal symmetries of $\mathcal{E}$ can only be taken into account by preliminarily embedding $\mathcal{E}$ in an auxiliary system $\mathcal{E}'$, of the form (3), and then searching for symmetries (higher or classical) of $\mathcal{E}'$. As a consequence, the nonlocal symmetry reduction of $\mathcal{E}$ comes from the local symmetry reduction of $\mathcal{E}'$. By applying proposition 1 to the distribution $\langle\widetilde{D}_x\rangle$ on $\mathcal{E}'$, one readily gets the following analogue of proposition 2:

Proposition 5. *Let $\{Y_1, \ldots, Y_{k+1}\}$ be a solvable structure for the one-dimensional distribution $\widetilde{\mathcal{D}} = \langle\widetilde{D}_x\rangle$ on the covering system $\mathcal{E}'$ of a k-order ODE $\mathcal{E}$. Then the distribution $\widetilde{\mathcal{D}}$ is integrable by quadratures and the general solution of $\mathcal{E}'$ can be obtained in an implicit form by subsequently integrating the system of 1-forms $\omega_{k+1}, \ldots, \omega_1$, in the given order.*

Moreover, since any covering system $\mathcal{E}'$ of the form (3) is naturally equipped with the volume form

$$\widetilde{\Omega} = dx \wedge du \wedge \ldots \wedge du_{k-1} \wedge dw,$$

one also has the following analog of proposition 3:

Proposition 6. *The vector fields $\{Y_1, \ldots, Y_{k+1}\}$ on $\mathcal{E}'$ determine a solvable structure for $\widetilde{\mathcal{D}} = \langle\widetilde{D}_x\rangle$ iff the following two conditions are satisfied:*

(a) $Y_1 \lrcorner Y_2 \lrcorner \ldots \lrcorner Y_{k+1} \lrcorner \widetilde{D}_x \lrcorner \widetilde{\Omega} \neq 0$;
(b) for any $s = 0, 1, \ldots, k$, *defining* $Y_0 = \widetilde{D}_x$, *one has*

$$L_{Y_{s+1}}(Y_j) \wedge Y_0 \wedge Y_1 \wedge \ldots \wedge Y_s = 0, \qquad \forall j = 0, 1, \ldots, s.$$

[3] We denote by $\langle Y_1, \ldots, Y_r\rangle_{\mathbb{R}}$ the $\mathbb{R}$-linear span of the fields $Y_1, \ldots, Y_r$.

Unfortunately, if one considers a general covering system $\mathcal{E}'$, it is not true in general that $\mathcal{E}'$ inherits the local symmetries of $\mathcal{E}$. Hence, if for example $\mathcal{E}$ has a solvable algebra $\mathcal{G}$ of local symmetries, by considering a generic covering $\mathcal{E}'$ one not only may lose the local symmetries belonging to $\mathcal{G}$ but also acquires one more dimension. It follows that, in this case, one should also determine a solvable structure with one more vector field.

In particular, when working with solvable symmetry algebras, the above facts raise the question of weather $\mathcal{E}'$ can be chosen in such a way that $\mathcal{G}$ is inherited and the dimensional growth is compensated by the presence of some new nonlocal symmetry. An answer to this question is provided by the following:

Proposition 7. *Let $\mathcal{E}$ be a k-order ODE of the form (1) which admits an algebra $\mathcal{G}$ of local symmetries. The algebra $\mathcal{G}$ is inherited by a covering system $\mathcal{E}'$, of the form (3), iff H is a joint scalar invariant of $\mathcal{G}$. In particular, if $\partial_w H = 0$, $\mathcal{G}$ can be extended to the Lie algebra $\widehat{\mathcal{G}} = \mathcal{G} \oplus \langle \partial_w \rangle_{\mathbb{R}}$. If in addition $\mathcal{G}$ is solvable, then also $\widehat{\mathcal{G}}$ is solvable.*

Proof. Each $X \in \mathcal{G}$ is a vector field tangent to $\mathcal{E}$ of the form $X = \overline{D}_x^i(\psi)\partial_{u_i}$. The proof of the first part readily follows by imposing the invariance of $\mathcal{E}'$ under the vector fields of $\mathcal{G}$. In fact, since each $X \in \mathcal{G}$ is such that $X(u_k - f)|_{\mathcal{E}^{(\infty)}} = 0$, one readily gets that $X(w_1 - H) = -X(H) = 0$. Hence, if H is a function of the joint scalar invariants of $\mathcal{G}$ and $\partial_w H = 0$, one gains also the new symmetry ∂_w. The thesis follows by observing that ∂_w commute with all the vector fields of $\mathcal{G}$. □

However, when $\partial_w H = 0$, the advantage of solvable structures on $\mathcal{E}'$ is that one does not necessarily need that the symmetry algebra $\mathcal{G}$ is inherited by $\mathcal{E}'$. In fact, since any $X \in \mathcal{G}$ is such that

$$[X, \overline{D}_x] = \alpha \overline{D}_x$$

for some α, then one readily gets that X is a symmetry of the distribution $\langle \widetilde{D}_x, \partial_w \rangle$. Hence, if $\mathcal{G} = \langle Y_1, Y_2, \ldots, Y_r \rangle_{\mathbb{R}}$, one could search just for solvable structures for $\langle \widetilde{D}_x \rangle$ which extend $\{\partial_w, Y_1, Y_2, \ldots, Y_r\}$ and possibly include nonlocal symmetries.

In particular all the above facts apply to the special case when $\mathcal{E}'$ is a λ-covering system for $\mathcal{E}$. In this case, if Y is a nonlocal symmetry corresponding to a λ-symmetry, one could search for solvable structures which extend for example the two-dimensional algebra spanned by ∂_w and Y.

5. Examples

The following examples provide an insight into the applications of the above reduction scheme. All the examples make use of solvable structures adapted to symmetries of the given ODE. In particular, since in a solvable structure there is no need to distinguish between symmetries and other vector fields, all these vector fields will be treated on the same footing.

Example 1. Consider the ODE

$$u_3 = \frac{u_2^2\left(u_1^2 - 2uu_2\right)}{u_1^4}. \tag{12}$$

The algebra of classical symmetries $\mathcal{G}_0$ of (12) is two-dimensional and generated by the functions

$$\varphi_1 = xu_1 - u, \qquad \varphi_2 = u_1.$$

Therefore classical symmetries do not suffice for the complete symmetry reduction of (12).

Remark 6. By reducing equation (12) through the symmetries X_2, X_1 generated by φ_2 and φ_1, in the given order, one first obtains the reduced equation $U_{\xi\xi} = (-U^{-3}+U^{-2})U_\xi^2 - 2\xi U^{-3}U_\xi^3$ and subsequently the Riccati equation $V_\eta = 1 - 2\eta^{-3}V^2 - \eta^{-1}V + \eta^{-2}V$. Here by $U = u_1, \xi = u$ we denote the basis invariants of X_2 and analogously by $V = \xi U_\xi$ and $\eta = U$ the basis invariants of X_1. The above Riccati equation can explicitly be integrated even though it is not possible to exhibit an explicit solution of the initial equation (12). Below we show an alternative integration of (12) by means of a solvable structure.

However, one can also search for higher symmetries. In this case, one can find that up to first order the algebra of symmetries is $\mathcal{G}_0 \oplus \mathcal{G}_1$ where $\mathcal{G}_1$ is a three-dimensional algebra generated by the functions

$$\varphi_3 = xu_1^2 - 2uu_1, \qquad \varphi_4 = u_1^2, \qquad \varphi_5 = \mathrm{e}^{1/u_1}u_1^2.$$

Let us denote by X_i the symmetries generated by the functions φ_i above. We have that the only nonvanishing commutators are

$$\begin{aligned} &[X_1, X_2] = X_2, \qquad &&[X_1, X_4] = X_4, \qquad &&[X_1, X_5] = X_5, \\ &[X_2, X_3] = -X_4, \qquad &&[X_3, X_5] = X_5. \end{aligned}$$

Therefore, X_1, X_2 and X_4 span a solvable algebra which can be used to completely reduce (12). This special example shows the advantage of considering also higher order symmetries in the reduction of ODE.

Here we will apply the solvable algebra $\{Y_1 = X_4, Y_2 = X_2, Y_3 = X_1\}$ in the integration of (12) through the subsequent integration of the forms ω_3, ω_2 and ω_1.

Now, by making use of (7) and the coordinate expressions of the fields Y_i and $Z = \overline{D}_x$, one gets $\Delta = \left(u_1^2 - 2uu_2\right)u_2^2$ and

$$\begin{aligned} \omega_3 &= \frac{2u_2}{u_1^2 - 2uu_2}\,\mathrm{d}u + \frac{2uu_2 - u_1^2 - 2u_1^3}{\left(u_1^2 - 2uu_2\right)u_1^2}\,\mathrm{d}u_1 + \frac{u_1^2}{\left(u_1^2 - 2uu_2\right)u_2}\,\mathrm{d}u_2, \\ \omega_2 &= \mathrm{d}x + \frac{2xu_2 - 2u_1}{u_1^2 - 2uu_2}\,\mathrm{d}u + \frac{(xu_1 - 2u)u_1}{\left(u_1^2 - 2uu_2\right)u_2}\,\mathrm{d}u_2 \\ &\quad + \frac{(2xuu_1 - 4u^2)u_2^2 + \left(-2xu_1^4 + (2u - x)u_1^3 + 2uu_1^2\right)u_2 + u_1^5}{\left(u_1^2 - 2uu_2\right)u_2u_1^3}\,\mathrm{d}u_1, \\ \omega_1 &= -\frac{1}{u_1^2 - 2uu_2}\mathrm{d}u + \frac{u_1^5 + uu_1^2u_2 - 2u^2u_2^2}{\left(u_1^2 - 2uu_2\right)u_1^4u_2}\,\mathrm{d}u_1 \\ &\quad - \frac{u}{\left(u_1^2 - 2uu_2\right)u_2}\mathrm{d}u_2. \end{aligned}$$

The 1-form ω_3 is exact and one can check that $\omega_3 = \mathrm{d}I_3$ with

$$I_3 = \frac{1}{u_1} + \ln\left(\frac{u_2}{2uu_2 - u_1^2}\right).$$

Then, by restricting ω_2 and ω_1 to the integral manifolds $\Sigma_3 := \{I_3 = c_3\}$, one finds

$$\begin{aligned} \omega_2|_{\Sigma_3} &= \mathrm{d}x + \frac{\mathrm{e}^{\frac{1+u_1\ln(u_2)-c_3u_1}{u_1}} - u_1^3}{u_1^3u_2}\mathrm{d}u_1 + \frac{u_1}{u_2^2}\mathrm{d}u_2, \\ \omega_1|_{\Sigma_3} &= \frac{\mathrm{e}^{\frac{1+u_1\ln(u_2)-c_3u_1}{u_1}}}{2u_1^4u_2}\mathrm{d}u_1 + \frac{1}{2u_2^2}\mathrm{d}u_2. \end{aligned}$$

In this case both $\omega_2|_{\Sigma_3}$ and $\omega_1|_{\Sigma_3}$ are exact 1-forms. This is due to the fact that both $\{Y_1, Y_2, Y_3\}$ and $\{Y_2, Y_1, Y_3\}$ are solvable structures. Then, when restricted to the level manifold Σ_3, ω_2 and ω_1 are the exterior derivatives of the functions

$$I_2 = x - \frac{u_1}{u_2} + \frac{(u_1 - 1)\,\mathrm{e}^{\frac{1-c_3u_1}{u_1}}}{u_1},$$

$$I_1 = -\frac{1}{2u_2} - \frac{1}{2}\frac{\left(1 - 2u_1 + 2u_1^2\right)\mathrm{e}^{\frac{1-c_3u_1}{u_1}}}{u_1^2},$$

respectively. It follows that the general solution of (12) can be written in the implicit form $\{I_1 = c_1, I_2 = c_2, I_3 = c_3\}$.

Example 2. Consider the ODE

$$u_2 = -\frac{x^2}{4u^3} - u - \frac{1}{2u}. \tag{13}$$

As shown in [21], (13) has no point symmetries but admits a λ-symmetry with $\lambda = x/u^2$. If we consider the system

$$\begin{cases} u_2 = -\dfrac{x^2}{4u^3} - u - \dfrac{1}{2u}, \\ w_1 = \dfrac{x}{u^2} \end{cases} \tag{14}$$

a nonlocal symmetry Y of (13) which corresponds to the λ-symmetry found by Muriel and Romero in [21] is that generated by the functions $\varphi^1 = u\mathrm{e}^w$ and $\varphi^2 = -2\mathrm{e}^w$. We will search for solvable structures which extend the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w\rangle$.

By making use of the Maple 11 routines, it is not difficult to find the most general solvable structure $\{Y_1, Y_2, Y_3\}$ for $\mathcal{D} = \langle Y_0\rangle$ (recall that, in this case, Y_0 is the total derivative operator restricted to (14)). The determining equations (10), in fact, now are $L_{Y_3}(Y_j) \wedge Y_0 \wedge Y_1 \wedge Y_2 = 0$, $j = 0, 1, 2$, and admit the general solution $Y_3 = a_1\partial_x + a_2\partial_u + a_3\partial_{u_1} + a_4\partial_w$, with

$$a_1 = \frac{\left(4u^4 + 4u^2u_1^2 + 4xuu_1 + x^2\right)F + 8u((uu_1 + x)a_2 - u^2a_3)}{8u^4 + u^2\left(8u_1^2 + 4\right) + 8xuu_1 + 2x^2},$$

a_2, a_3, a_4 arbitrary functions of (x, u, u_1, w) and F an arbitrary function of $x + \arctan((2uu_1 + x)/(2u^2))$.

Nevertheless, since here we are only interested in the integration of (13), we just consider the following particular solution (with $F = 2$, $a_2 = a_4 = 0$ and $a_3 = -1/(2u)$)

$$Y_3 = \partial_x - \frac{1}{2u}\partial_{u_1}.$$

In this case, since both $\{Y_1, Y_2, Y_3\}$ and $\{Y_1, Y_3, Y_2\}$ are solvable structures, one can check that $\mathrm{d}\omega_3 = \mathrm{d}\omega_2 = 0$. Hence, by proceeding as in the previous example, one finds $\omega_3 = \mathrm{d}I_3$, $\omega_2 = \mathrm{d}I_2$ with

$$I_2 = 2\ln u - w - \ln\left(4u^2u_1^2 + 4xuu_1 + 4u^4 + x^2\right),$$

$$I_3 = \arctan\left(\frac{u^2u_1 + xu/2}{u^3}\right) + x.$$

It follows that, when restricted to the level manifold $\{I_2 = c_2, I_3 = c_3\}$, ω_1 is an exact 1-form and one can check that it is the exterior derivative of the function

$$I_1 = \frac{2\,\mathrm{e}^{c_2}u^2}{\cos(x - c_3)^2} + 2\,\mathrm{e}^{c_2}\ln(\cos(c_3 - x)) - 2\,\mathrm{e}^{c_2}x\tan(c_3 - x).$$

Then, one gets the general solution of (14) in the implicit form $\{I_1 = c_1, I_2 = c_2, I_3 = c_3\}$. In this case, however, one can solve with respect to u and gets the solution of (13) in the form

$$u = \pm\cos(x - c_2)\sqrt{\frac{c_1\,\mathrm{e}^{-c_3}}{2} - x\tan(x - c_2) - \ln(\cos(x - c_2))}$$

This solution depends on three arbitrary constants, but of course one of them is inessential. In fact, by suitably rearranging c_1 one can write the general solution of (13) in the form

$$u = \pm\cos(x - C_2)\sqrt{C_1 - x\tan(x - C_2) - \ln(\cos(x - C_2))}, \qquad C_1, C_2 \in \mathbb{R}.$$

This solution coincides with that found in [21].

Example 3. Consider the ODE

$$u_2 = \frac{u_1}{(1+x)x} - \frac{x^2 + x^3}{4(1+x)u^3} - \frac{x}{2(1+x)u}. \tag{15}$$

It can be shown that (15) has no point symmetries but admits a λ-covering with $\lambda = x/u^2$. Hence, if we consider the covering system $\mathcal{E}'$ defined by (15) together with $w_1 = \lambda$, it admits the nonlocal symmetry Y generated by $\varphi^1 = u\mathrm{e}^w$ and $\varphi^2 = -2\,\mathrm{e}^w$. We will search for solvable structures which extend the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w\rangle$.

In contrast to the previous example, in this case it is more difficult to find the most general solution of the determining equations (10). Nevertheless, one can find some particular solutions. In this case we just consider the following particular solution:

$$Y_3 = \frac{(1+x)(x + 2uu_1)^2}{8u_1(x + uu_1)}\partial_u + \frac{x + x^2 + 2uu_1 + 2xuu_1}{4u^2(x + uu_1)}\partial_w.$$

As in the previous example, since both $\{Y_1, Y_2, Y_3\}$ and $\{Y_1, Y_3, Y_2\}$ are solvable structures, one can check that $\mathrm{d}\omega_3 = \mathrm{d}\omega_2 = 0$. Then, by proceeding as in the previous examples, one gets the general solution of (15):

$$u = \pm(C - 2x + 2\ln(1+x))\left(\int \frac{-x}{(C - 2x + 2\ln(1+x))^2}\mathrm{d}x\right)^{1/2}, \tag{16}$$

with $C \in \mathbb{R}$. Note that, as in the previous example, in the general integral of the covering system $\mathcal{E}'$ there was a third inessential constant we have gauged out in (16). Moreover, we have absorbed one of the arbitrary constants in the indefinite integral.

Example 4. Consider the ODE

$$u_2 = (xu_1 - xu^2 + u^2)\,\mathrm{e}^{-1/u} + \frac{2u_1^2}{u} + u_1. \tag{17}$$

It can be shown that (17) has no point symmetries but admits a λ-covering with $\lambda = x\mathrm{e}^{-1/u} - 1/x$. Hence, if we consider the covering system $\mathcal{E}'$ defined by (17) together with $w_1 = \lambda$, it admits the nonlocal symmetry Y generated by $\varphi^1 = x\mathrm{e}^w$ and $\varphi^2 = x\mathrm{e}^w$. We will search for solvable structures which extend the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w\rangle$.

Also in this case, it is very complicated to find the most general solution of the determining equations (10). Nevertheless, one can again find some particular solutions. In this case we just consider the following particular solution:

$$Y_3 = u^2\,\mathrm{e}^x\,\partial_{u_1}.$$

In contrast to the previous examples, $\{Y_1, Y_3, Y_2\}$ is not a solvable structure, hence in this case only ω_3 is closed. By proceeding as in the previous examples, one still can find the general solution of the covering system $\mathcal{E}'$. Then, by solving with respect to u one gets the general solution of (17):

$$u = \frac{1}{C\,\mathrm{e}^x + \ln\left(\int \frac{-x}{\mathrm{e}^{C\mathrm{e}^x}}\,\mathrm{d}x\right)}, \tag{18}$$

with $C \in \mathbb{R}$. Here, as in the previous examples, we have gauged out from (18) all inessential arbitrary constants.

Example 5. Consider the ODE

$$u_2 = \frac{2u_1^2}{u} + \left(x\,\mathrm{e}^{x/u} - \frac{4}{x}\right)u_1 - \left(\frac{3u^2}{x} + u\right)\mathrm{e}^{x/u} + xu^2 + \frac{2u}{x^2}. \tag{19}$$

It can be shown that (19) has no point symmetries but admits a λ-covering with $\lambda = x\,\mathrm{e}^{x/u}$. Hence, if we consider the covering system $\mathcal{E}'$ defined by (19) together with $w_1 = \lambda$, it admits the nonlocal symmetry Y generated by $\varphi^1 = \mathrm{e}^w u^2/x$ and $\varphi^2 = -\mathrm{e}^w$. We will search for solvable structures which extend the non-Abelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w\rangle$.

Also in this case, it is very complicated to find the most general solution of the determining equations (10). Nevertheless, one can again find some particular solutions. In this case we just consider the following particular solution:

$$Y_3 = \frac{u^2}{x^3}\partial_{u_1}.$$

By proceeding as in the previous examples, one still can find the general solution of the covering system $\mathcal{E}'$. Then, by solving with respect to u one gets the general solution of (19):

$$u = \frac{-x^2}{C + \frac{x^5}{20} + x\ln\left(\int \frac{-x}{\mathrm{e}^{\frac{C}{x}+\frac{x^4}{20}}}\mathrm{d}x\right)}, \tag{20}$$

with $C \in \mathbb{R}$. Here, as in the previous examples, we have gauged out from (20) all inessential arbitrary constants.

6. Concluding remarks

Solvable structures for a one-dimensional distribution $\mathcal{D}$ can be determined by solving the system of equations (10). The analysis of these determining equations is in general more feasible if one makes use of symbolic manipulation packages like those written for Maple V distributions. In particular, we suggest the packages DifferentialGeometry and Jets by I Anderson (and coworkers) and M Marvan, respectively. The first package is recommended for various computations involved in the determination of solvable structures and the further integration of the forms ω_i. The second package is very helpful in the computation of symmetries and the analysis of differential consequences of a given system of equations.

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